Exam Geometry WIMTK-08

January 20, 2015

Note: Usage of Do Carmo's textbook is allowed. Give a precise reference to the theory you use for solving the problems. You may *not* use the result of any of the exercises.

Problem 1 (5+5+7+5+8=30 pts.)
Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with Frenet-Serret frame $\{t(s), n(s), b(s)\}$ at $\alpha(s)$. The curvature and torsion at $\alpha(s)$ are denoted by $k(s)$ and $\tau(s)$, respectively. The *Darboux vector* $\omega(s)$ of the curve at $\alpha(s)$ is given by

$$\omega(s) = \tau(s)t(s) - k(s)b(s).$$

1. Show that the Frenet-Serret identities are equivalent with

$$\begin{aligned} t' &= t \wedge \omega, \\ n' &= n \wedge \omega, \\ b' &= b \wedge \omega. \end{aligned}$$

2. Show that the Darboux vector is constant if and only if the curvature and the torsion are constant.

Let $\nu(s)$ be the unit vector field along α defined by $\nu = \beta^{-1}\omega$, with $\beta = \sqrt{k^2 + \tau^2}$. Let $u(s) = \nu(s) \wedge n(s)$, then $\{u(s), \nu(s), n(s)\}$ is an orthonormal moving frame. From now on assume that k and τ are constant.

3. Show that the moving frame $\{u(s), \nu(s), n(s)\}$ satisfies the identities

$$\begin{aligned} u' &= \beta n, \\ n' &= -\beta u, \\ \nu' &= 0. \end{aligned}$$

Let $u_0 = u(0), n_0 = n(0)$ and $\nu_0 = \nu(0)$.

4. Show that $u(s) = \cos(\beta s)\, u_0 + \sin(\beta s)\, n_0$. Express $n(s)$ and $\nu(s)$ in terms of β, s and u_0, n_0, ν_0.

5. Determine an expression for $\alpha(s)$ in terms of of $\alpha(0)$, k, τ, s and u_0, n_0, ν_0, and show that the curve lies on a circular cylinder with radius $r := k\beta^{-2}$, the axis of which is the line with direction ν through the point $\alpha(0) + rn_0$.
(Hint: express the unit vector $t(s)$ in terms of the frame $\{u(s), \nu(s), n(s)\}$.)

Problem 2 (20+10 = 30 pts.)
Let $x = x(u, v)$ be a regular parameterised surface. Define the *offset surface* by

$$y(u, v) = x(u, v) + \varepsilon N(u, v),$$

where $\varepsilon > 0$ is a small constant. For the solution of this exercise, you may want to consider the matrix (a_{ij}) that represents dN_p in the basis $\{x_u, x_v\}$, however you should have no need to explicitly compute (or look up expressions for) the coefficients of this matrix.

1. Show that
$$y_u \wedge y_v = (1 - 2\varepsilon H + \varepsilon^2 K) x_u \wedge x_v,$$
where H and K are the mean and Gaussian curvatures of x.

2. Assume that ε is small enough that $(1 - 2\varepsilon H + \varepsilon^2 K) > 0$. Let $\tilde{K}$ and $\tilde{H}$ be the Gaussian and mean curvatures of y. Show that
$$\tilde{K} = \frac{K}{1 - 2\varepsilon H + \varepsilon^2 K}$$
and
$$\tilde{H} = \frac{H - \varepsilon K}{1 - 2\varepsilon H + \varepsilon^2 K}.$$
Hint: What is the expression for $\tilde{N}$ in terms of $\{N, x_u, x_v\}$?

Problem 3 (10 + 12 + 8 = 30 pts.)
Let C be a regular curve (without self-intersections) in the half-plane $\{(x, 0, z) \mid x > 0\}$. Let S be the surface of revolution in $\mathbb{R}^3$ obtained by rotating C about the z-axis. It is convenient to use use the parameterisation

$$x(u, v) = (f(v) \cos u, f(v) \sin u, g(v)),$$

such that $v \mapsto (f(v), 0, g(v))$ is an arc-length parameterisation of C.

1. Let p and q be two points on C, and let $w_p \in T_pS$ be a unit tangent vector making an angle φ_0 with (the tangent line of) C at p. Let w_q be the vector obtained by parallel transporting w_p from p to q along C. Prove that w_q also makes an angle φ_0 with this meridian.

Consider a point $p \in C$. Let Γ be the parallel circle of S through p. Let $\overline{w}_p \in T_pS$ be the vector obtained by parallel transporting w_p once around Γ. Let ϑ_0 be the angle between T_pS and the horizontal plane through p.

2. Express the angle $\Delta\varphi$ between w_p and $\overline{w}_p$ in terms of ϑ_0.

3. Prove that $\Delta\varphi$ is zero if and only if Γ is a geodesic of S.